1

Def: (a) $H_0 \in \mathfrak{h}_{\mathbb{R}}$ is called regular if $\alpha(H_0) \neq 0 \quad \forall \alpha \in \Delta$
$\exists$ such a H_0

(b) $\alpha \leq \beta$ if $\alpha(H_0) \leq \beta(H_0)$ ~ only partial ordering.

either $\alpha(H_0) > 0$ or $\alpha(H_0) < 0$

(c) $\Delta^+ := \{\alpha \in \Delta \mid \alpha(H_0) > 0\}$ Pick H_0 & Fixed.

(d) $\alpha \in \Delta^+$ is called simple/fundamental if $\alpha \neq \beta + \gamma$ with $\beta, \gamma \in \Delta^+$

(e) $F := \{\alpha \in \Delta^+, \alpha \text{ simple}\}$ is the set of simple/fundamental roots
It is called a base in Serre.

(f) Cartan matrix $(n_{\beta\alpha})_{\alpha,\beta \in F}$ $\quad n_{\alpha\alpha} = 2.$ $\quad \begin{pmatrix} n_{\alpha\beta} \leq 0 \\ \text{if } \alpha \neq \beta. \end{pmatrix}$

(g) Dynkin diagram: Given $\{\alpha_i\} = F$.
Its Dynkin diagram is constructed as
(i) —O— for each α_i
draw a circle
(ii) Connect α_i, α_j if $n_{ij} \neq 0$
by $\underline{n_{ij} \cdot n_{ji}}$ many lines $\in \{1, 2, 3\}$
} Coxeter graph.
(iii) If $n_{ij} \cdot n_{ji} > 1$, mark the circle with the number to indicate the ratio of the relative length.
(Humphreys: $\Rightarrow$, $\Rrightarrow$ to indicate the shorter one)

Without (iii) it is called a Coxter graph.

Zillen Prop 4.19
Also see Humphreys 10.4.
Needs some results on reflections.

Main therrem: [which we shall NOT prove]

Theorem: (a) $\boxed{\mathfrak{g} \text{ is simple iff } D(\mathfrak{g}) \text{ is connected}}$

(b) $D(\mathfrak{g})$ is independent of the choice of $\mathfrak{h}$ & ordering
{ namely H_0 then Δ^+

(c) $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ iff $D(\mathfrak{g}_1) = D(\mathfrak{g}_2)$

(d) Every one of the 9 possibilities arises from a simple Lie algebra.

First, the partial ordering via a regular element $H_0 \in \eta_{\mathbb{R}}$ will provide a way to pick a (base) via F.

in Serre & Humphreys.

<u>Lemma</u>. ⓐ $B(\alpha_i, \alpha_j) \le 0 \quad \forall \alpha_i, \alpha_j \in F \quad \alpha_i \ne \alpha_j$

ⓑ F is linearly indep.

ⓒ $\forall \alpha \in \Delta^+$ then $\underline{\alpha = \sum n_j \alpha_j}$ $\quad \alpha_j \in F$ & $n_j \ge 0$ are integers

In particular F provides a basis for Δ. [Serre called a base ~~Humphreys~~]

<u>Pf</u>: ⓐ. If $B(\alpha_i, \alpha_j) > 0 \Rightarrow \alpha_i - \alpha_j \in \Delta$ $\left[\begin{matrix} \beta + n\alpha & -p \le n \le q \\ \beta - n_{\alpha\beta}\alpha \in \Delta & n_{\alpha\beta} = p - q \\ & -n_{\alpha\beta} = -p+q \end{matrix} \right]$

$\Rightarrow$ Either $\underbrace{\alpha_i}_{=\beta} - \underbrace{\alpha_j}_{=\alpha} \in \Delta^+$ or $\alpha_j - \alpha_i \in \Delta^+$

Then $\alpha_j = \alpha_i + (\alpha_j - \alpha_i)$

or $\alpha_i = \alpha_j + (\alpha_i - \alpha_j)$ $\quad \Longrightarrow$ one of them (α_i, α_j) is NOT simple

linearly dependent

ⓑ If $\Rightarrow \sum p_i \alpha_i - \sum q_j \alpha_j = 0$ $\quad$ Collecting $p_i > 0$, $q_j > 0$

$\alpha_i \ne \alpha_j$

$\Rightarrow \sum p_i \alpha_i = \sum q_j \alpha_j$

$B(\sum p_i \alpha_i, \sum q_j \alpha_j) \ge 0 \quad \Longrightarrow \sum p_i \alpha_i = \sum q_j \alpha_j = 0$

But $\quad \| \quad \sum p_i q_j \underbrace{B(\alpha_i, \alpha_j)}_{\le 0}$ $\quad$ & $B(\alpha_i, \alpha_j) = 0$

& $B(\sum p_i \alpha_i, \sum p_i \alpha_i) = 0$

$\Rightarrow \sum p_i \alpha_i = 0$

But $0 = \langle \underbrace{\sum p_i \alpha_i}_{0}, H_0 \rangle = \sum \underbrace{p_i}_{>0} \underbrace{\alpha_i(H_0)}_{>0} \Longrightarrow$ A Contradiction

For ⓒ, we argue by contradiction. If $S := \{ \alpha \in \Delta_+ \mid \alpha$ can not be written as $\sum n_i \alpha_i \quad n_i \in \mathbb{Z}_+$ & $\alpha_i \in F \}$

Then pick $\alpha \in S$ such that $\alpha(H_0)$ is the smallest.

Since α is Not in F $\Rightarrow$ $\alpha = \beta_1 + \beta_2 \quad \beta_i \in \Delta^+$

But $\alpha(H_0) = \underset{>0}{\beta_1(H_0)} + \underset{>0}{\beta_2(H_0)}$. $\Rightarrow$ $\beta_i(H_0) < \alpha(H_0)$ $\; i=1,2$

Then $\beta_i = \sum n^i_k \alpha^i_k \qquad n^i_k \in \mathbb{Z}_+ \;\; \& \;\; \alpha_k \in F$

This proves that $\alpha = \sum n^1_k \alpha^1_k + n^2_k \alpha^2_k$. $\Rightarrow (\Rightarrow\Leftarrow)$

(2)

$$n_{\alpha\beta} = 2\frac{B(\beta,\alpha)}{B(\alpha,\alpha)} = p - q$$

$$n_{\beta\alpha} = 2\frac{B(\beta,\alpha)}{B(\beta,\beta)}$$

Table: Page 63 of Ziller (Table 4.13).

$r = 4\cos^2\angle(H_\alpha H_\beta)$	$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\angle(H_\alpha\, H_\beta)$	Relative Size	α string containing β / at least contains
0	0	0	$\frac{\pi}{2}$	N/A	N/A
1	1	1	$\frac{\pi}{3}$	$\lvert\beta\rvert^2 = \lvert\alpha\rvert^2$	$\beta,\ \beta-\alpha$
1	-1	-1	$\frac{2\pi}{3}$	$\lvert\beta\rvert^2 = \lvert\alpha\rvert^2$	$\beta,\ \beta+\alpha$
2	2	1	$\frac{\pi}{4}$	$\lvert\beta\rvert^2 = 2\lvert\alpha\rvert^2$	$\beta\ \ \beta-\alpha\ \ \beta-2\alpha$
2	-2	-1	$3\frac{\pi}{4}$	$\lvert\beta\rvert^2 = 2\lvert\alpha\rvert^2$	$\beta\ \ \beta+\alpha,\ \beta+2\alpha$
3	3	1	$\frac{\pi}{6}$	$\lvert\beta\rvert^2 = 3\lvert\alpha\rvert^2$	$\beta,\ \beta-\alpha,\ \beta-2\alpha,\ \beta-3\alpha$
3	-3	-1	$\frac{5\pi}{6}$	$\lvert\beta\rvert^2 = 3\lvert\alpha\rvert^2$	$\beta,\ \beta+\alpha,\ \beta+2\alpha,\ \beta+3\alpha$

$|\alpha|^2 = B(\alpha,\alpha)$

We use prop 4.10 (a)

may be longer

maximal length

$n_{\alpha\beta} = \varepsilon r \qquad \frac{|\beta|^2}{|\alpha|^2} = r, \qquad n_{\beta\alpha} = \varepsilon$

$\cos\varphi = \frac{\varepsilon}{2}\sqrt{r}$

Can not have string longer than 4

$$\beta,\ \beta+\alpha\ \ \underline{\beta+2\alpha}\ \ \beta+3\alpha\ \ \beta+4\alpha$$

$$\beta'-2\alpha\ \ \beta'-\alpha\ \ \beta'\ \ \beta'+\alpha\ \ \beta'+2\alpha$$

Then $n_{\alpha\beta} = (p-q) = -4$

$n_{\beta\alpha} = \text{integer} \neq 0$

$\Rightarrow \; 4 \leq |n_{\alpha\beta} \cdot n_{\beta\alpha}| = 4\cos^2\angle(H_\alpha H_\beta) \leq 4$

$\Rightarrow H_\alpha \parallel H_\beta$

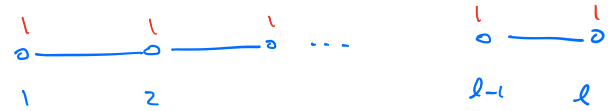
Namely $\beta = \pm\alpha \Rightarrow (\Rightarrow\Leftarrow)$.

This implies a classification of the Dynkin diagram.

<u>Coxeter graph</u> : the graph without (iii) marking.

Theorem. if Δ is irreducible of rank ℓ. then its Dynkin diagram is one of following

A_ℓ $(\ell \geq 1)$

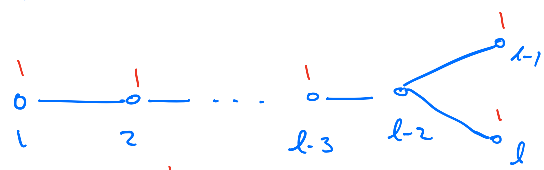


B_ℓ $(\ell \geq 2)$

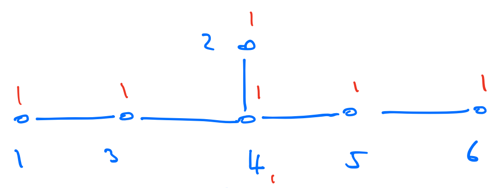
in terms of Coxeter graph only 1--

C_ℓ $(\ell \geq 3)$

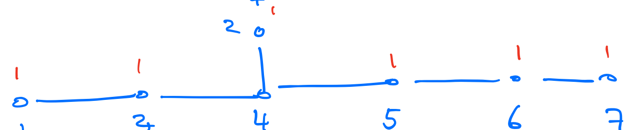
D_ℓ $(\ell \geq 4)$



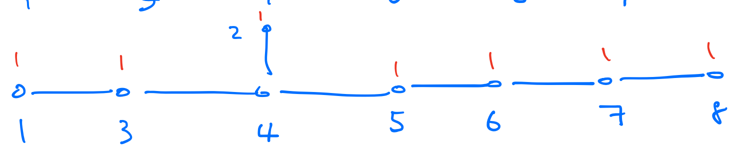
E_6



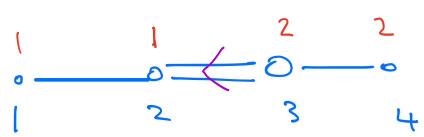
E_7



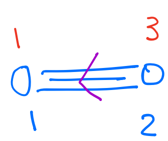
E_8



F_4



G_2



The Lie algebra lists of last lecture is due to this classification theorem.

At least the labelings are from the above.

The rest is to check. (i) A, B, C, D can be realized by the four classical Lie algebras. — Can be done directly.

(ii) Identify & Construct the simple exceptional Lie algebras & the Compact exceptional Lie groups. (Corresponding to E, F, G type)

(3) The proof of the classification result. (We focus on the Coxeter graph)

We divide it into the following 10 steps. (As in Humphreys)

For the step 10. $\varepsilon = \sum_{i=1}^{p-1} i\,\varepsilon_i$

$$\|\varepsilon\|^2 = \sum_{i=1}^{p-1} i^2 - \sum_{i=1}^{p-2} i(i+1) = (p-1)^2 - \frac{(p-1)(p-2)}{2} = \frac{(p-1)}{2}\left[2p-2-p+2\right] = \frac{p(p-1)}{2}$$

$$\|\xi\|^2 = \frac{r(r-1)}{2}, \quad \|\eta\|^2 = \frac{q(q-1)}{2}$$

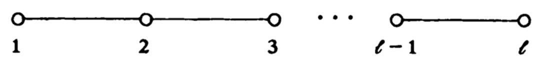
$\{\varepsilon_i\}, \varepsilon. \quad \perp \quad \{\xi_i\}.\ \xi \quad \& \quad \perp \ \{\eta_i\}.\ \eta.$

$$\frac{(\psi, \varepsilon)^2}{\|\psi\|^2\|\varepsilon\|^2} = \frac{(p-1)^2}{\frac{(p-1)p}{2}}\underbrace{(\psi, \varepsilon_{p-1})^2}_{=\frac{1}{4}} = \frac{2}{4}\left(1-\frac{1}{p}\right) = \frac{1}{2}\left(1-\frac{1}{p}\right)$$

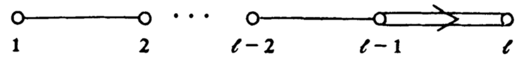
$$\frac{1}{2}\left(1-\frac{1}{p}\right) + \frac{1}{2}\left(1-\frac{1}{q}\right) + \frac{1}{2}\left(1-\frac{1}{r}\right) < 1 \iff \frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1.$$

This limits the possibility subtantially.

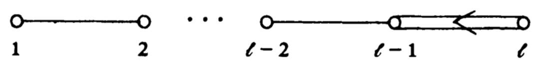
A_ℓ ($\ell \geq 1$):



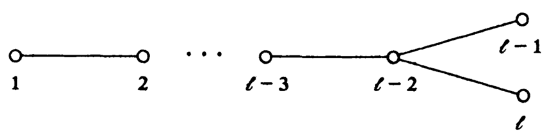
B_ℓ ($\ell \geq 2$):



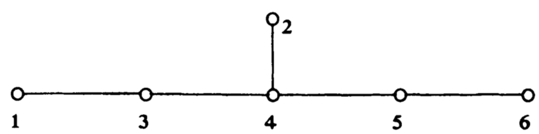
C_ℓ ($\ell \geq 3$):



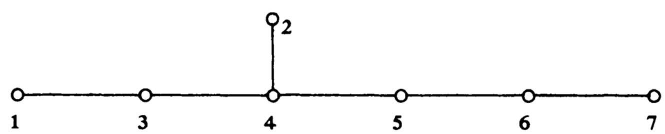
D_ℓ ($\ell \geq 4$):



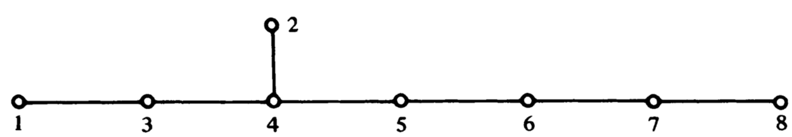
E_6:



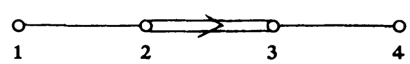
E_7:



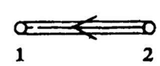
E_8:



F_4:



G_2:



The restrictions on ℓ for types $A_\ell - D_\ell$ are imposed in order to avoid duplication. Relative to the indicated numbering of simple roots, the corresponding Cartan matrices are given in Table 1. Inspection of the diagrams listed above reveals that in all cases except B_ℓ, C_ℓ, the Dynkin diagram can be deduced from the Coxeter graph. However, B_ℓ and C_ℓ both come from a single Coxeter graph, and differ in the relative numbers of short and long simple roots. (These root systems are actually dual to each other, cf. Exercise 5.)

Proof of Theorem. The idea of the proof is to classify first the possible Coxeter graphs (ignoring relative lengths of roots), then see what Dynkin diagrams result. Therefore, we shall merely apply some elementary euclidean geometry to finite sets of vectors whose pairwise angles are those prescribed by the Coxeter graph. Since we are ignoring lengths, it is easier to work for the time being with sets of unit vectors. For maximum flexibility, we make

Table 1. *Cartan matrices*

$$
\mathsf{A}_\ell:\quad
\begin{pmatrix}
2 & -1 & 0 & & & \cdot & \cdot & \cdot & & 0 \\
-1 & 2 & -1 & 0 & & \cdot & \cdot & \cdot & & 0 \\
0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2
\end{pmatrix}
$$

$$
\mathsf{B}_\ell:\quad
\begin{pmatrix}
2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\
-1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -2 \\
0 & 0 & 0 & & \cdot & \cdot & \cdot & 0 & -1 & 2
\end{pmatrix}
$$

$$
\mathsf{C}_\ell:\quad
\begin{pmatrix}
2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\
-1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\
0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -1 \\
0 & 0 & & & \cdot & \cdot & \cdot & 0 & -2 & 2
\end{pmatrix}
$$

$$
\mathsf{D}_\ell:\quad
\begin{pmatrix}
2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\
-1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & & \cdot & \cdot & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & & \cdot & \cdot & & -1 & 2 & -1 & -1 \\
0 & 0 & & \cdot & \cdot & & 0 & -1 & 2 & 0 \\
0 & 0 & & \cdot & \cdot & & 0 & -1 & 0 & 2
\end{pmatrix}
$$

$$
\mathsf{E}_6:\quad
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
$$

$$
\mathsf{E}_7:\quad
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
$$

$$
\mathsf{E}_8:\quad
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
$$

$$
\mathsf{F}_4:\quad
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}.
\qquad
\mathsf{G}_2:\quad
\begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}
$$

ε_i being $\frac{\alpha_i}{|\alpha_i|}$

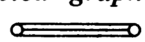
only the following assumptions: E is a euclidean space (of arbitrary dimension), $\mathfrak{A} = \{\varepsilon_1, \ldots, \varepsilon_n\}$ is a set of n linearly independent unit vectors which satisfy $(\varepsilon_i, \varepsilon_j) \leq 0$ $(i \neq j)$ and $4(\varepsilon_i, \varepsilon_j)^2 = 0, 1, 2,$ or 3 $(i \neq j)$. Such a set of vectors is called (for brevity) **admissible**. (*Example*: Elements of a base for a root system, each divided by its length.) We attach a graph Γ to the set $\mathfrak{A}$ just as we did above to the simple roots in a root system, with vertices i and j $(i \neq j)$ joined by $4(\varepsilon_i, \varepsilon_j)^2$ edges. Now our task is to determine all the connected graphs associated with admissible sets of vectors (these include all connected Coxeter graphs). This we do in steps, the first of which is obvious. (Γ is not assumed to be connected until later on.)

(1) *If some of the ε_i are discarded, the remaining ones still form an admissible set, whose graph is obtained from Γ by omitting the corresponding vertices and all incident edges.*

(2) *The number of pairs of vertices in Γ connected by at least one edge is strictly less than n.* Set $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$. Since the ε_i are linearly independent, $\varepsilon \neq 0$. So $0 < (\varepsilon, \varepsilon) = n + 2\sum_{i<j} (\varepsilon_i, \varepsilon_j)$. Let i, j be a pair of (distinct) indices for which $(\varepsilon_i, \varepsilon_j) \neq 0$ (i.e., let vertices i and j be joined). Then $4(\varepsilon_i, \varepsilon_j)^2 = 1, 2,$ or 3, so in particular $2(\varepsilon_i, \varepsilon_j) \leq -1$. In view of the above inequality, the number of such pairs cannot exceed $n-1$.

(3) *Γ contains no cycles.* A cycle would be the graph Γ' of an admissible subset $\mathfrak{A}'$ of $\mathfrak{A}$ (cf. (1)), and then Γ' would violate (2), with n replaced by Card $\mathfrak{A}'$.

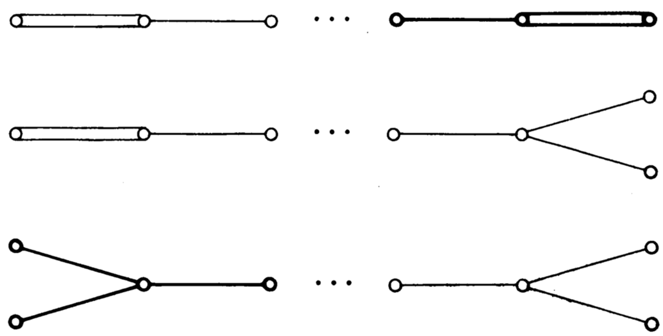
(4) *No more than three edges can originate at a given vertex of Γ.* Say $\varepsilon \in \mathfrak{A}$, and $\eta_1, \ldots, \eta_k$ are the vectors in $\mathfrak{A}$ connected to ε (by 1, 2, or 3 edges each), i.e., $(\varepsilon, \eta_i) < 0$ with $\varepsilon, \eta_1, \ldots, \eta_k$ all distinct. In view of (3), no two η's can be connected, so $(\eta_i, \eta_j) = 0$ for $i \neq j$. Because $\mathfrak{A}$ is linearly independent, some unit vector η_0 in the span of $\varepsilon, \eta_1, \ldots, \eta_k$ is orthogonal to $\eta_1, \ldots, \eta_k$; clearly $(\varepsilon, \eta_0) \neq 0$ for such η_0. Now $\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i)\eta_i$, so $1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2$. This forces $\sum_{i=1}^{k} (\varepsilon, \eta_i)^2 < 1$, or $\sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2 < 4$. But $4(\varepsilon, \eta_i)^2$ is the number of edges joining ε to η_i in Γ.

(5) *The only connected graph Γ of an admissible set $\mathfrak{A}$ which can contain a triple edge is*  *(the Coxeter graph* G_2*). This follows at once from (4).*

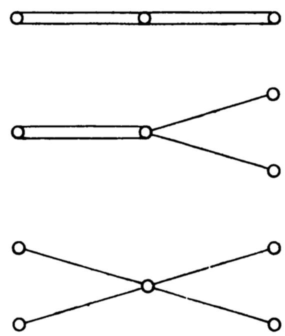
(6) *Let $\{\varepsilon_1, \ldots, \varepsilon_k\} \subset \mathfrak{A}$ have subgraph*  *(a simple chain in Γ. If $\mathfrak{A}' = (\mathfrak{A} - \{\varepsilon_1, \ldots, \varepsilon_k\}) \cup \{\varepsilon\}$, $\varepsilon = \sum_{i=1}^{k} \varepsilon_i$, then $\mathfrak{A}'$ is admissible.* (The graph of $\mathfrak{A}'$ is obtained from Γ by shrinking the simple chain to a point.) Linear independence of $\mathfrak{A}'$ is obvious. By hypothesis, $2(\varepsilon_i, \varepsilon_{i+1}) = -1$ $(1 \leq i \leq k-1)$, so $(\varepsilon, \varepsilon) = k + 2\sum_{i<j} (\varepsilon_i, \varepsilon_j) = k - (k-1) = 1$. So ε is a unit vector. Any $\eta \in \mathfrak{A} - \{\varepsilon_1, \ldots, \varepsilon_k\}$ can be connected to at most one of $\varepsilon_1, \ldots, \varepsilon_k$ (by (3)), so $(\eta, \varepsilon) = 0$ or else $(\eta, \varepsilon) = (\eta, \varepsilon_i)$ for $1 \leq i \leq k$. In either case, $4(\eta, \varepsilon)^2 = 0, 1, 2,$ or 3.

$2(\varepsilon_1\ \varepsilon_2) + 2(\varepsilon_2\ \varepsilon_3) + \ldots$

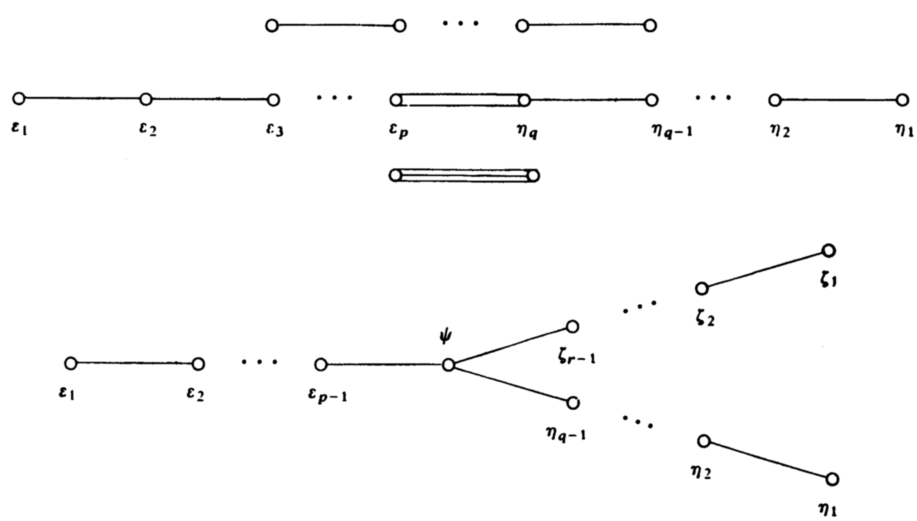
(7) Γ *contains no subgraph of the form*:

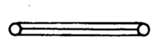


Suppose one of these graphs occurred in Γ; by (1) it would be the graph of an admissible set. But (6) allows us to replace the simple chain in each case by a single vertex, yielding (respectively) the following graphs which violate (4):

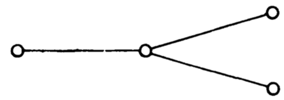


(8) *Any connected graph* Γ *of an admissible set has one of the following forms*:

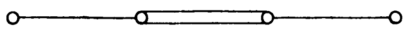
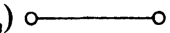
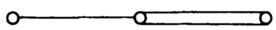


Indeed, only  contains a triple edge, by (5). A connected graph containing more than one double edge would contain a subgraph

which (7) forbids, so at most one double edge occurs. Moreover, if Γ has a double edge, it cannot also have a "node" (branch point)



(again by (7)), so the second graph pictured is the only possibility (cycles being forbidden by (3)). Finally, let Γ have only single edges; if Γ has no node, it must be a simple chain (again because no cycles are allowed). It cannot contain more than one node (7), so the fourth graph is the only remaining possibility.

(9) *The only connected Γ of the second type in* (8) *is the Coxeter graph* F_4  *or the Coxeter graph* $\mathsf{B}_n(=\mathsf{C}_n)$  $\cdots$ .

Set $\varepsilon = \sum_{i=1}^{p} i\varepsilon_i$, $\eta = \sum_{i=1}^{q} i\eta_i$. By hypothesis, $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_i, \eta_{i+1})$, and other pairs are orthogonal, so $(\varepsilon, \varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = p(p+1)/2$, $(\eta, \eta) = q(q+1)/2$. Since $4(\varepsilon_p, \eta_q)^2 = 2$, we also have $(\varepsilon, \eta)^2 = p^2q^2(\varepsilon_p, \eta_q)^2 = p^2q^2/2$. The Schwartz inequality implies (since ε, η are obviously independent) ~~that~~ $(\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta)$, or $p^2q^2/2 < p(p+1)q(q+1)/4$, whence $(p-1)(q-1) < 2$. The possibilities are: $p = q = 2$ (whence F_4) or $p = 1$ (q arbitrary), $q = 1$ (p arbitrary).

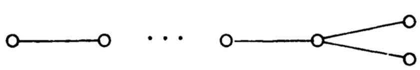
$= p^2 - \frac{p(p-1)}{2}$

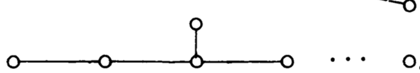
$= \frac{p}{2}[2p-(p-1)]$

$(p^2+p)(q^2+q)$ $\frac{p^2q^2}{4} < \frac{p^2q+q^2p+pq}{4}$

$pq < p+q+1$

(10) *The only connected Γ of the fourth type in* (8) *is the Coxeter graph* D_n

 *or the Coxeter graph* E_n ($n = 6$, 7 *or* 8)

. Set $\varepsilon = \sum i\varepsilon_i$, $\eta = \sum i\eta_i$, $\zeta = \sum i\zeta_i$. It is clear that ε, η, ζ are mutually orthogonal, linearly independent vectors, and that ψ is not in their span. As in the proof of (4) we therefore obtain $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 < 1$, where θ_1, θ_2, θ_3 are the respective angles between ψ and ε, η, ζ. The same calculation as in (9), with $p-1$ in place of p, shows that $(\varepsilon, \varepsilon) = p(p-1)/2$, and similarly for η, ζ. Therefore $\cos^2\theta_1 = (\varepsilon, \psi)^2/(\varepsilon, \varepsilon)(\psi, \psi) = (p-1)^2(\varepsilon_{p-1}, \psi)^2/(\varepsilon, \varepsilon) = \frac{1}{4}(2(p-1)^2/p(p-1)) = (p-1)/2p = \frac{1}{2}(1-1/p)$. Similarly for θ_2, θ_3. Adding, we get the inequality $\frac{1}{2}(1-1/p+1-1/q+1-1/r) < 1$, or (*) $1/p+1/q+1/r > 1$. (This inequality, by the way, has a long mathematical history.) By changing labels we may assume that $1/p \le 1/q \le 1/r$ $(\le 1/2$; if p, q, or r equals 1, we are back in type A_n). In particular, the inequality (*) implies $3/2 \ge 3/r > 1$, so $r = 2$. Then $1/p + 1/q > 1/2$, $2/q > 1/2$, and $2 \le q < 4$. If $q = 3$, then $1/p > 1/6$ and necessarily $p < 6$. So the possible triples (p, q, r) turn out to be: $(p, 2, 2) = \mathsf{D}_n$; $(3, 3, 2) = \mathsf{E}_6$; $(4, 3, 2) = \mathsf{E}_7$; $(5, 3, 2) = \mathsf{E}_8$.

The preceding argument shows that the connected graphs of admissible sets of vectors in euclidean space are all to be found among the Coxeter graphs of types A–G. In particular, the Coxeter graph of a root system must be of one of these types. But in all cases except B_ℓ, C_ℓ, the Coxeter graph

$\frac{1}{2}+\frac{1}{3}+\boxed{} > 1$ $p = 3.\ 4.\ 5$

Not many choice